

Supersymmetry in the $O(N)$ Gross-Neveu Models?

Herbert Morales*

Escuela de Física

Universidad de Costa Rica

San José, Costa Rica

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Abstract

We study a special classical current in the $O(3)$ Gross-Neveu model that becomes supersymmetric when quantum anomalies are included. Following its definition, we generalize the current for the general case, the $O(N)$ Gross-Neveu models. We compute its algebra and discuss the possibility of supersymmetry can be established for these models.

*Electronic address: hmorales@fisica.ucr.ac.cr

I. INTRODUCTION

The $O(N)$ Gross-Neveu model (GNM) is a renormalizable field theory of an N -component Majorana Fermi field, transforming in the fundamental representation of the orthogonal group $O(N)$, with a quartic self-interaction. This model was first introduced by Nambu and Jona-Lasinio in four dimensions as a dynamical model of elementary particles in which nucleons and mesons are derived from a fundamental spinor field [1]. In two dimensions, Gross and Neveu studied the large N limit and performed an expansion in powers of $1/N$ to all orders in the coupling constant [2]. Thus, they found that the model displays dynamical symmetry breakdown that generates, in the resulting theory, a fermion mass and dimensional transmutation, i.e the conversion of a dimensionless coupling constant into a mass scale parameter. All these properties follow from the fact that the $O(N)$ GNM is an asymptotically free theory. Subsequently, other features of the model have been found. In a semiclassical analysis, the particle spectrum of GNM reveals a very rich structure [3]. The integrability of the classical model due to the existence of an infinite number of conservation laws [4] (which survive quantization) has been used to compute the exact S -matrix of the theory [5], [6].

The Lagrangian for the model is

$$\mathcal{L} = \frac{i}{2} \bar{\psi}_i \not{\partial} \psi_i + g(\bar{\psi}_i \psi_i)^2, \quad (1)$$

where ψ_i is a Majorana spinor, g is a dimensionless coupling constant and repeated indices are summed over. A fermion mass term is forbidden by the discrete chiral symmetry

$$\psi_i \rightarrow \gamma^5 \psi_i. \quad (2)$$

If we use Dirac spinors in place of Majorana ones, the symmetry group will be $U(N)$. In either case, the symmetry that suffers spontaneous breakdown is the discrete chiral symmetry.

In the Majorana-Weyl representation, the Lagrangian in (1) can be written in terms of chiral components of the spinors as ($j \neq i$)

$$\mathcal{L} = i\psi_{iR}\partial_+\psi_{iR} + i\psi_{iL}\partial_-\psi_{iL} + 4g\psi_{iL}\psi_{jL}\psi_{iR}\psi_{jR}, \quad (3)$$

and so can the field equations

$$\partial_+\psi_{iR} = 4ig\psi_{jL}\psi_{jR}\psi_{iL}, \quad \partial_-\psi_{iL} = -4ig\psi_{jL}\psi_{jR}\psi_{iR}. \quad (4)$$

In this paper, we investigate if the $O(N)$ GNM can have supercurrents at the quantum level. This work is organized as follows. In section II, we define and study a special classical current in the $O(3)$ GNM that becomes supersymmetric when quantum anomalies are included. With its modified current, we compute the superalgebra. In section III, we generalize the current definition to the $O(N)$ GNM and compute the new algebra. Then we conclude with remarks on the supersymmetric interpretation of this current and its algebra.

II. THE $O(3)$ GROSS-NEVEU MODEL

Following [7], let us define the following fermionic currents which will later become the supercurrents

$$j^+ \equiv \psi_{1R}\psi_{2R}\psi_{3R}, \quad j^- \equiv \psi_{1L}\psi_{2L}\psi_{3L}. \quad (5)$$

Classically, they are conserved

$$\partial_+ j^+ = 0, \quad \partial_- j^- = 0. \quad (6)$$

These results follow from the field equations (4) and $\psi_{iR}^2 = \psi_{iL}^2 = 0$.

Strictly, j^+ and j^- form the lower and upper components of the spinor current, $J^\mu = \frac{1}{2}(\bar{\psi}_1\gamma_\nu\psi_2)\gamma^\nu\gamma^\mu\gamma^5\psi_3$, i.e.

$$J^0 = \begin{pmatrix} -j^- \\ j^+ \end{pmatrix}, \quad J^1 = \begin{pmatrix} j^- \\ j^+ \end{pmatrix}, \quad (7)$$

so that (6) is implemented in $\partial_\mu J^\mu = 0$.

At the quantum level, these conservation laws are modified by anomalies. From dimensional and symmetry considerations, the modifications must take the form

$$\partial_\mu J^\mu + f(g)\partial_\mu R^\mu = 0, \quad (8)$$

where $R^\mu = \frac{1}{2}(\bar{\psi}_1\gamma_\nu\psi_2)\gamma^\mu\gamma^\nu\gamma^5\psi_3 + (\bar{\psi}_1\gamma^5\psi_2)\gamma^\mu\psi_3$, i.e.

$$R^0 = \begin{pmatrix} -r^+ \\ r^- \end{pmatrix}, \quad R^1 = \begin{pmatrix} -r^+ \\ -r^- \end{pmatrix}, \quad (9)$$

$$\begin{aligned} r^+ &= \psi_{1R}\psi_{2R}\psi_{3L} + \psi_{3R}\psi_{1R}\psi_{2L} + \psi_{2R}\psi_{3R}\psi_{1L}, \\ r^- &= \psi_{1L}\psi_{2L}\psi_{3R} + \psi_{3L}\psi_{1L}\psi_{2R} + \psi_{2L}\psi_{3L}\psi_{1R}, \end{aligned} \quad (10)$$

and $f(g)$ is some function of the coupling constant. (See [7] for a complete discussion of this result.)

Now we define the modified current of J^μ which is quantumly conserved by (8)

$$\mathcal{J}^\mu \equiv E(g)(J^\mu + f(g)R^\mu), \quad (11)$$

where $E(g)$ is a normalization coefficient. The spinor charge associated with this current is defined in the usual way

$$Q \equiv \int dx \mathcal{J}^0 = E(g) \int dx (J^0 + f(g)R^0), \quad (12)$$

or in components, $Q = (-Q^- \ Q^+)^T$,

$$Q^\pm = E(g) \int dx (j^\pm + f(g)r^\mp). \quad (13)$$

Using contractions (see Appendix B), we compute the algebra made by these charges

$$\begin{aligned} (Q^+)^2 &= \frac{E^2(g)}{2\pi} (1 - f^2(g)) \int dx \left(\frac{i}{2} \psi_{iR} \partial_1 \psi_{iR} + \frac{\pi f(g)}{1 + f(g)} \psi_{iL} \psi_{jL} \psi_{iR} \psi_{jR} \right), \\ (Q^-)^2 &= \frac{E^2(g)}{2\pi} (1 - f^2(g)) \int dx \left(\frac{-i}{2} \psi_{iL} \partial_1 \psi_{iL} + \frac{\pi f(g)}{1 + f(g)} \psi_{iL} \psi_{jL} \psi_{iR} \psi_{jR} \right), \\ \{Q^+, Q^-\} &= 0. \end{aligned} \quad (14)$$

(We have omitted a c-number in the first two equations.) For this algebra to be supersymmetric, we need the integrals to be proportional to the conserved charges associated with translations, $P^\pm = P^0 \pm P^1 = \int dx (T^{00} \pm T^{01})$, respectively. Therefore we read off

$$\begin{aligned} T^{00} &= \frac{E^2(g)}{4\pi} (f^2(g) - 1) \left(\frac{i}{2} (\psi_{iL} \partial_1 \psi_{iL} - \psi_{iR} \partial_1 \psi_{iR}) - 2\pi \left(\frac{f(g)}{1 + f(g)} \right) \psi_{iL} \psi_{jL} \psi_{iR} \psi_{jR} \right), \\ T^{01} &= \frac{E^2(g)}{4\pi} (f^2(g) - 1) \left(\frac{-i}{2} (\psi_{iL} \partial_1 \psi_{iL} + \psi_{iR} \partial_1 \psi_{iR}) \right). \end{aligned} \quad (15)$$

These components of $T^{\mu\nu}$ are expected to be the quantum ones. (Notice that the expressions in the big parentheses agree with the classical components if $f(g)/(1 + f(g)) = 2g/\pi$.) Now $E(g)$ can be found through the proper normalization of $T^{\mu\nu}$ and $f(g)$ will encode the trace anomaly, $T^\mu_\mu \neq 0$. However, for our purpose we do not need to determine the exact form of these functions of g (see next paragraph). The conservation of this stress-energy tensor follows from supersymmetry.

Moreover, the third equation in (14) says that the central charge for the model is zero (independent of the coupling constant). The exact form of $f(g)$ is just important for the trace anomaly, but not for the central charge. (See [8] for a discussion of the equivalence between this model and the supersymmetric sine-Gordon model.)

III. THE $O(N)$ GROSS-NEVEU MODEL

As the above section, let us define the following fermionic currents for $N > 3$

$$j_{ijk}^+ \equiv \psi_{iR}\psi_{jR}\psi_{kR}, \quad j_{ijk}^- \equiv \psi_{iL}\psi_{jL}\psi_{kL}, \quad (16)$$

where normal ordering should be understood on the right-hand side of these equations. A more proper definition for these currents can be established by using the $O(N)$ totally antisymmetric tensor

$$\begin{aligned} s_{i_1 i_2 \dots i_{N-3}}^+ &\equiv \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} \psi_{i_{N-2}R} \psi_{i_{N-1}R} \psi_{i_N R} = \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} j_{i_{N-2} i_{N-1} i_N}^+, \\ s_{i_1 i_2 \dots i_{N-3}}^- &\equiv \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} \psi_{i_{N-2}L} \psi_{i_{N-1}L} \psi_{i_N L} = \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} j_{i_{N-2} i_{N-1} i_N}^-. \end{aligned} \quad (17)$$

Now the interpretation of j_{ijk}^\pm is easier: these currents form the components of a $O(N)$ antisymmetric pseudotensor of rank $N - 3$ (a pseudovector for $N = 4$ and a pseudoscalar for $N = 3$) and there are $N(N - 1)(N - 2)/6$ independent nonzero tensor components (or currents). A similar interpretation follows for the currents j_{ijk}^\pm . However, we would rather use j_{ijk}^\pm than $s_{i_1 i_2 \dots i_{N-3}}^\pm$ because the index notation is simpler.

Classically, they are not conserved

$$\begin{aligned} \partial_+ j_{ijk}^+ &= 4ig \psi_{iL} \psi_{lR} r_{ijk}^+, \\ \partial_- j_{ijk}^- &= -4ig \psi_{iL} \psi_{lR} r_{ijk}^-, \end{aligned} \quad (18)$$

where the sum over l does not include the values of i, j and k and the “anomalies” are

$$\begin{aligned} r_{ijk}^+ &= \psi_{iR}\psi_{jR}\psi_{kL} + \psi_{kR}\psi_{iR}\psi_{jL} + \psi_{jR}\psi_{kR}\psi_{iL}, \\ r_{ijk}^- &= \psi_{iL}\psi_{jL}\psi_{kR} + \psi_{kL}\psi_{iL}\psi_{jR} + \psi_{jL}\psi_{kL}\psi_{iR}. \end{aligned} \quad (19)$$

The formal definition for these anomalies is given by

$$\begin{aligned} t_{i_1 i_2 \dots i_{N-3}}^+ &\equiv \frac{1}{2!} \epsilon_{i_1 i_2 \dots i_N} \psi_{i_{N-2}R} \psi_{i_{N-1}R} \psi_{i_N L} = \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} r_{i_{N-2} i_{N-1} i_N}^+, \\ t_{i_1 i_2 \dots i_{N-3}}^- &\equiv \frac{1}{2!} \epsilon_{i_1 i_2 \dots i_N} \psi_{i_{N-2}L} \psi_{i_{N-1}L} \psi_{i_N R} = \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} r_{i_{N-2} i_{N-1} i_N}^-. \end{aligned} \quad (20)$$

Strictly, j_{ijk}^+ and j_{ijk}^- form the lower and upper components of the spinor current, $J_{ijk}^\mu = \frac{1}{2}(\bar{\psi}_i \gamma_\nu \psi_j) \gamma^\nu \gamma^\mu \gamma^5 \psi_k$, i.e.

$$J_{ijk}^0 = \begin{pmatrix} -j_{ijk}^- \\ j_{ijk}^+ \end{pmatrix}, \quad J_{ijk}^1 = \begin{pmatrix} j_{ijk}^- \\ j_{ijk}^+ \end{pmatrix}, \quad (21)$$

and so do r_{ijk}^+ and r_{ijk}^- for the spinor anomaly, $R_{ijk}^\mu = \frac{1}{2}(\bar{\psi}_i \gamma_\nu \psi_j) \gamma^\mu \gamma^\nu \gamma^5 \psi_k + (\bar{\psi}_i \gamma^5 \psi_j) \gamma^\mu \psi_k$, i.e.

$$R_{ijk}^0 = \begin{pmatrix} -r_{ijk}^+ \\ r_{ijk}^- \end{pmatrix}, \quad R_{ijk}^1 = \begin{pmatrix} -r_{ijk}^+ \\ -r_{ijk}^- \end{pmatrix}. \quad (22)$$

so that (18) is implemented in $\partial_\mu J_{ijk}^\mu + 2ig (\bar{\psi}_l \psi_l) \gamma_\mu R_{ijk}^\mu = 0$.

At the quantum level, we redefine these currents with their anomalies as

$$\mathcal{J}_{ijk}^\mu \equiv E_N (J_{ijk}^\mu + f_N R_{ijk}^\mu), \quad (23)$$

where $f_N = f_N(g)$ is some function of the coupling constant and $E_N = E_N(g)$ is a normalization coefficient. Obviously, different N implies different f_N and E_N , but these functions do not change for fixed N and g , due to their $O(N)$ invariance.

The spinor charges associated with these currents are defined by

$$Q_{ijk} \equiv \int dx \mathcal{J}_{ijk}^0 = E_N \int dx (J_{ijk}^0 + f_N R_{ijk}^0), \quad (24)$$

or in components, $Q_{ijk} = (-Q_{ijk}^- \ Q_{ijk}^+)^T$,

$$Q_{ijk}^\pm = E_N \int dx (j_{ijk}^\pm + f_N r_{ijk}^\mp). \quad (25)$$

However, we can define formally these spinor charges with

$$\mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^\pm \equiv E_N \int dx (s_{i_1 i_2 \dots i_{N-3}}^\pm + f_N t_{i_1 i_2 \dots i_{N-3}}^\mp) = \frac{1}{3!} \epsilon_{i_1 i_2 \dots i_N} Q_{i_{N-2} i_{N-1} i_N}^\pm. \quad (26)$$

Using the Q_{ijk}^\pm algebra (see Appendix B), we compute the algebra made by the contracted $\mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^\pm$

$$\begin{aligned} \mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^+ \mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^+ &= \frac{(N-1)!}{4\pi} E_N^2 (1 - f_N^2) \int dx \left(\frac{i}{2} \psi_{iR} \partial_1 \psi_{iR} + \right. \\ &\quad \left. \frac{2\pi}{N-1} \left(\frac{f_N}{1 + f_N} \right) \psi_{iL} \psi_{jL} \psi_{iR} \psi_{jR} \right), \\ \mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^- \mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^- &= \frac{(N-1)!}{4\pi} E_N^2 (1 - f_N^2) \int dx \left(\frac{-i}{2} \psi_{iL} \partial_1 \psi_{iL} + \right. \\ &\quad \left. \frac{2\pi}{N-1} \left(\frac{f_N}{1 + f_N} \right) \psi_{iL} \psi_{jL} \psi_{iR} \psi_{jR} \right), \\ \{\mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^+, \mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^-\} &= 0. \end{aligned} \quad (27)$$

(We have again omitted a c-number in the first two equations.) For this algebra to be “supersymmetric”, we require the integrals to be proportional to the conserved charges, P^\pm ,

respectively. Therefore we read off

$$\begin{aligned} T^{00} &= \frac{(N-1)!}{8\pi} E_N^2 (f_N^2 - 1) \left(\frac{i}{2} (\psi_{iL} \partial_1 \psi_{iL} - \psi_{iR} \partial_1 \psi_{iR}) - \frac{4\pi}{N-1} \left(\frac{f_N}{1+f_N} \right) \psi_{iL} \psi_{jL} \psi_{iR} \psi_{jR} \right), \\ T^{01} &= \frac{(N-1)!}{8\pi} E_N^2 (f_N^2 - 1) \left(\frac{-i}{2} (\psi_{iL} \partial_1 \psi_{iL} + \psi_{iR} \partial_1 \psi_{iR}) \right). \end{aligned} \quad (28)$$

We try a similar interpretation of these results as those in the above section: these components of $T^{\mu\nu}$ are expected to be the quantum ones. (Notice again that the classical expressions can be obtained if $f_N/(1+f_N)(N-1) = g/\pi$.) Consequently, E_N can be found through the proper normalization of $T^{\mu\nu}$ and f_N will encode the trace anomaly. If we consider the third equation in (27) as usual, we conclude that the central charges for the $O(N)$ Gross-Neveu models are zero and independent of the coupling constant g , the exact forms of E_N and f_N are not important for finding the “central charges”. The conservation of this stress-energy tensor cannot follow from “supersymmetry”, since we have not shown the current conservation for \mathcal{J}_{ijk}^μ . We expect that the current conservation occurs only for some values of g , because dimensional and symmetry considerations do not allow us to constraint or infer it. Thus, we shall have to explore the currents with more details in the future.

As has been seen, our proposal for “supersymmetry” requires to contract the charges $\mathcal{Q}_{i_1 i_2 \dots i_{N-3}}^\pm$, otherwise their algebra is very complicated (see Appendix B). This requirement does not allow us to make a direct identification with (extended) supersymmetry. Further research of this algebra is needed to understand its implications and supersymmetry connections. Obviously, one can start it with the free models ($g = 0$), but even though the expressions are simpler, the contraction requirement is still needed.

Appendix A: Notation

We use the metric $\eta^{00} = -\eta^{11} = 1$ and the antisymmetric tensor $\epsilon_{01} = -\epsilon^{01} = 1$. In light-cone coordinates, we define $x^\pm \equiv x^0 \pm x^1$, thus $\eta_{+-} = 1/2$, $\eta^{+-} = 2$ and $\epsilon_{+-} = -1/2$, $\epsilon^{+-} = 2$. Moreover, $\partial_\pm \equiv (\partial_0 \pm \partial_1)/2$.

For the two-dimensional Dirac algebra, we define $\gamma^5 = \gamma_5 \equiv \frac{1}{2} \epsilon_{\mu\nu} \gamma^\mu \gamma^\nu$. Some useful properties of the γ matrices are $\gamma^\mu \gamma^5 = \epsilon^{\mu\nu} \gamma_\nu$ and $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} I - \epsilon^{\mu\nu} \gamma^5$. In the Majorana-Weyl representation, the γ matrices are written as $\gamma^0 = \sigma_2$, $\gamma^1 = -i\sigma_1$ and $\gamma^5 = -\sigma_3$. In light-cone coordinates, we have $\gamma^\pm \equiv \gamma^0 \pm \gamma^1$. Furthermore, a Majorana spinor is written

in chiral components as $\psi = (\psi_L \ \psi_R)^T$, with $\psi_L^\dagger = \psi_L$ and $\psi_R^\dagger = \psi_R$. And a Dirac spinor is given by two Majorana spinors, $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$.

Some useful delta-function properties are $\delta^{(n)}(x - y) = (-1)^n \delta^{(n)}(y - x)$ and

$$\int dx f(x) \delta^{(n)}(x - y) = (-1)^n \partial^n f(y), \quad (\text{A1})$$

all derivatives are respect to the argument.

Appendix B: Fermi fields

The contraction of free massless Dirac fields is given by the Wightman function (see [9], [10])

$$S^{(+)}(\xi = y - x) \equiv \overline{\psi(y)}\psi(x) = \frac{1}{2\pi} \int \frac{dk^1}{2k^0} k^\mu \gamma_\mu e^{-ik \cdot \xi - k^0 \epsilon}, \quad (\text{B1})$$

where $\epsilon > 0$ is a UV regularization.

We define

$$C^\mu(\xi) \equiv \frac{1}{2} \text{tr}(\gamma^\mu S^{(+)}(\xi)) = \frac{1}{4\pi} \int dk^1 \text{sgn}(k^\mu) e^{-ik \cdot \xi - k^0 \epsilon}, \quad (\text{B2})$$

so that $S^{(+)}(\xi) = C^\mu(\xi) \gamma_\mu$.

In the Majorana-Weyl representation, we have

$$\begin{aligned} C^+(\xi = y - x) &= \overline{\psi_R(y)}\psi_R(x) = -\frac{i}{2\pi} \left(\frac{1}{\xi^- - i\epsilon} \right), \\ C^-(\xi = y - x) &= \overline{\psi_L(y)}\psi_L(x) = -\frac{i}{2\pi} \left(\frac{1}{\xi^+ - i\epsilon} \right), \\ \overline{\psi_R(y)}\psi_L(x) &= \overline{\psi_L(y)}\psi_R(x) = 0, \end{aligned} \quad (\text{B3})$$

where ψ_R and ψ_L are the chiral components of a Majorana spinor.

For equal-time situations (from here onward x denotes just the space coordinate), we have the following properties

$$\begin{aligned} C^\pm(-x) &= C^\mp(x), \\ (C^\pm(x))^n &= \left(\mp \frac{i}{2\pi} \right)^{n-1} \frac{1}{(n-1)!} \left(\partial_x^{(n-1)} C^\pm(x) \right), \\ (C^\pm(x))^n + (-1)^{n-1} (C^\pm(-x))^n &= \left(\mp \frac{i}{2\pi} \right)^{n-1} \frac{1}{(n-1)!} \left(\partial_x^{(n-1)} \delta(x) \right), \\ C^-(x)C^+(x) - C^-(-x)C^+(-x) &= 0, \\ (C^\pm(x))^2 C^\mp(x) + (C^\pm(-x))^2 C^\mp(-x) &= -\frac{1}{8\pi^2} \frac{\delta'(x)}{x}. \end{aligned} \quad (\text{B4})$$

Using these functions C^\pm and Wick's theorem, we can “derive” the equal-time canonical anticommutation relations for the Majorana-Weyl spinor components

$$\begin{aligned}\{\psi_R(x), \psi_R(y)\} &= C^+(x-y) + C^+(y-x) = \delta(x-y), \\ \{\psi_L(x), \psi_L(y)\} &= C^-(x-y) + C^-(y-x) = \delta(x-y), \\ \{\psi_R(x), \psi_L(y)\} &= 0.\end{aligned}\tag{B5}$$

Similarly, we can compute other equal-time commutators and anticommutators of these components. For example, we find

$$\begin{aligned}\left[(\psi_{1R}\psi_{2R})(x), (\psi_{1R}\psi_{2R})(y)\right] &= -(C^+(x-y))^2 + (C^+(y-x))^2 \\ &= \frac{i}{2\pi}\delta'(x-y), \\ \left[(\psi_{1L}\psi_{2L})(x), (\psi_{1L}\psi_{2L})(y)\right] &= -\frac{i}{2\pi}\delta'(x-y), \\ \left[(\psi_{1R}\psi_{2R})(x), (\psi_{1L}\psi_{2L})(y)\right] &= 0.\end{aligned}\tag{B6}$$

With the above results and

$$j^0 = i(\psi_{1R}\psi_{2R} + \psi_{1L}\psi_{2L}), \quad j^1 = i(\psi_{1R}\psi_{2R} - \psi_{1L}\psi_{2L}),\tag{B7}$$

we can obtain the nonzero current commutator

$$[j^0(x), j^1(y)] = -\frac{i}{\pi}\delta'(x-y),\tag{B8}$$

where the right-hand side is a Schwinger term [11]. One can also show the Kac-Moody algebra made by the fermion currents $j_{ij}^+ = 2i\psi_{iR}\psi_{jR}$ and $j_{ij}^- = 2i\psi_{iL}\psi_{jL}$ [12].

For finding the anticommutators of Q_{ijk}^\pm , we need the equal-time anticommutators of three-spinor products. The calculations are done in the same fashion as before, so that normal ordering should be understood on the right-hand side of the following equations. There is no sum over any index except for l and m which run only for the values of i, j, k and all indices are different from each other.

Hence, the nontrivial three-spinor anticommutators (those that involve more than one

contraction) needed for Q_{ijk}^+ are

$$\begin{aligned}
\{(\psi_{iR}\psi_{jR}\psi_{kR})(x), (\psi_{iR}\psi_{jR}\psi_{kR})(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{lR}(x)\psi_{lR}(y) + \frac{1}{8\pi^2}\delta''(x-y), \\
\{(\psi_{iR}\psi_{jR}\psi_{kR})(x), (\psi_{iR}\psi_{jR}\psi_{k'R})(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{kR}(x)\psi_{k'R}(y), \\
\{(\psi_{iR}\psi_{jL}\psi_{kL})(x), (\psi_{iR}\psi_{jL}\psi_{kL})(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{iR}(x)\psi_{iR}(y) + \frac{1}{8\pi^2}\frac{\delta'(x-y)}{x-y}, \\
\{(\psi_{iR}\psi_{jL}\psi_{kL})(x), (\psi_{iR}\psi_{jL}\psi_{k'L})(y)\} &= 0, \\
\{(\psi_{iR}\psi_{jL}\psi_{kL})(x), (\psi_{i'R}\psi_{jL}\psi_{kL})(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{iR}(x)\psi_{i'R}(y).
\end{aligned} \tag{B9}$$

From these results, we can easily obtain the anticommutators between j_{ijk}^+ 's

$$\begin{aligned}
\{j_{ijk}^+(x), j_{ijk}^+(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{lR}(x)\psi_{lR}(y) + \frac{1}{8\pi^2}\delta''(x-y), \\
\{j_{ijk}^+(x), j_{ij'k'}^+(y)\} &= \frac{i}{2\pi}\delta'(x-y)\psi_{kR}(x)\psi_{k'R}(y), \\
\{j_{ijk}^+(x), j_{ij'k'}^+(y)\} &= \delta(x-y)(\psi_{jR}\psi_{kR})(x)(\psi_{j'R}\psi_{k'R})(y), \\
\{j_{ijk}^+(x), j_{i'j'k'}^+(y)\} &= 0,
\end{aligned} \tag{B10}$$

between j_{ijk}^+ and r_{ijk}^-

$$\begin{aligned}
\{j_{ijk}^+(x), r_{ijk}^-(y)\} &= \frac{1}{2}\delta(x-y)(\psi_{lL}\psi_{mL})(y)(\psi_{lR}\psi_{mR})(x), \\
\{j_{ijk}^+(x), r_{ij'k'}^-(y)\} &= \delta(x-y)\left((\psi_{jR}\psi_{kR})(x)(\psi_{j'L}\psi_{k'L})(y) + (\psi_{iR}\psi_{kR})(x)(\psi_{i'L}\psi_{k'L})(y)\right), \\
\{j_{ijk}^+(x), r_{ij'k'}^-(y)\} &= \delta(x-y)(\psi_{jR}\psi_{kR})(x)(\psi_{j'L}\psi_{k'L})(y), \\
\{j_{ijk}^+(x), r_{i'j'k'}^-(y)\} &= 0,
\end{aligned} \tag{B11}$$

and between r_{ijk}^- 's

$$\begin{aligned}
\{r_{ijk}^-(x), r_{ijk}^-(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{lR}(x)\psi_{lR}(y) - \delta(x-y)(\psi_{lL}\psi_{mR})(x)(\psi_{mL}\psi_{lR})(y) + \\
&\quad + \frac{3}{8\pi^2}\frac{\delta'(x-y)}{x-y}, \\
\{r_{ijk}^-(x), r_{ij'k'}^-(y)\} &= -\frac{i}{2\pi}\delta'(x-y)\psi_{kR}(x)\psi_{k'R}(y) - \delta(x-y)\left((\psi_{iR}\psi_{kL})(x)(\psi_{k'R}\psi_{iL})(y) + \right. \\
&\quad + (\psi_{jR}\psi_{kL})(x)(\psi_{k'R}\psi_{jL})(y) + (\psi_{kR}\psi_{iL})(x)(\psi_{iR}\psi_{k'L})(y) + \\
&\quad \left. + (\psi_{kR}\psi_{jL})(x)(\psi_{jR}\psi_{k'L})(y)\right), \\
\{r_{ijk}^-(x), r_{ij'k'}^-(y)\} &= \delta(x-y)\left((\psi_{jL}\psi_{kL})(x)(\psi_{j'L}\psi_{k'L})(y) + \right. \\
&\quad \left. + (\psi_{jR}\psi_{kL} - \psi_{kR}\psi_{jL})(x)(\psi_{j'R}\psi_{k'L} - \psi_{k'R}\psi_{j'L})(y)\right), \\
\{r_{ijk}^-(x), r_{i'j'k'}^-(y)\} &= 0.
\end{aligned} \tag{B12}$$

Consequently, the algebra for Q_{ijk}^+ is given by

$$\begin{aligned}
\{Q_{ijk}^+, Q_{ijk}^+\} &= E_N^2(1-f_N) \int dx \left(\frac{i}{2\pi}(1+f_N) \psi_{lR} \partial_1 \psi_{lR} + f_N \psi_{lL} \psi_{mL} \psi_{lR} \psi_{mR} \right) + \text{c-}\#, \\
\{Q_{ijk}^+, Q_{ij'k'}^+\} &= E_N^2(1-f_N) \int dx \left(\frac{i}{4\pi}(1+f_N) (\psi_{kR} \partial_1 \psi_{k'R} + \psi_{k'R} \partial_1 \psi_{kR}) + \right. \\
&\quad \left. -f_N (\psi_{iL} \psi_{iR} + \psi_{jL} \psi_{jR}) (\psi_{kL} \psi_{k'R} + \psi_{k'L} \psi_{kR}) \right), \\
\{Q_{ijk}^+, Q_{ij'k'}^+\} &= E_N^2 \int dx \left(\psi_{jR} \psi_{kR} \psi_{j'R} \psi_{k'R} + f_N (\psi_{jL} \psi_{kL} \psi_{j'R} \psi_{k'R} + \psi_{j'L} \psi_{k'L} \psi_{jR} \psi_{kR}) + \right. \\
&\quad \left. f_N^2 (\psi_{jL} \psi_{kL} \psi_{j'L} \psi_{k'L} + (\psi_{jL} \psi_{kR} - \psi_{kL} \psi_{jR}) (\psi_{j'L} \psi_{k'R} - \psi_{k'L} \psi_{j'R})) \right), \\
\{Q_{ijk}^+, Q_{i'j'k'}^+\} &= 0.
\end{aligned} \tag{B13}$$

In similar fashion, we construct the Q_{ijk}^- algebra

$$\begin{aligned}
\{Q_{ijk}^-, Q_{ijk}^-\} &= E_N^2(1-f_N) \int dx \left(\frac{-i}{2\pi}(1+f_N) \psi_{lL} \partial_1 \psi_{lL} + f_N \psi_{lL} \psi_{mL} \psi_{lR} \psi_{mR} \right) + \text{c-}\#, \\
\{Q_{ijk}^-, Q_{ij'k'}^-\} &= E_N^2(1-f_N) \int dx \left(\frac{-i}{4\pi}(1+f_N) (\psi_{kL} \partial_1 \psi_{k'L} + \psi_{k'L} \partial_1 \psi_{kL}) + \right. \\
&\quad \left. -f_N (\psi_{iL} \psi_{iR} + \psi_{jL} \psi_{jR}) (\psi_{kL} \psi_{k'R} + \psi_{k'L} \psi_{kR}) \right), \\
\{Q_{ijk}^-, Q_{ij'k'}^-\} &= E_N^2 \int dx \left(\psi_{jL} \psi_{kL} \psi_{j'L} \psi_{k'L} + f_N (\psi_{jL} \psi_{kL} \psi_{j'R} \psi_{k'R} + \psi_{j'L} \psi_{k'L} \psi_{jR} \psi_{kR}) + \right. \\
&\quad \left. f_N^2 (\psi_{jR} \psi_{kR} \psi_{j'R} \psi_{k'R} + (\psi_{jL} \psi_{kR} - \psi_{kL} \psi_{jR}) (\psi_{j'L} \psi_{k'R} - \psi_{k'L} \psi_{j'R})) \right), \\
\{Q_{ijk}^-, Q_{i'j'k'}^-\} &= 0.
\end{aligned} \tag{B14}$$

Finally, we close the Q_{ijk}^\pm algebra with

$$\begin{aligned}
\{Q_{ijk}^+, Q_{ijk}^-\} &= 0, \\
\{Q_{ijk}^+, Q_{ij'k'}^-\} &= E_N^2 f_N (1-f_N) \int dx \left((\psi_{iL} \psi_{iR} + \psi_{jL} \psi_{jR}) (\psi_{kL} \psi_{k'L} + \psi_{kR} \psi_{k'R}) \right), \\
\{Q_{ijk}^+, Q_{ij'k'}^-\} &= E_N^2 f_N \int dx \left((\psi_{jR} \psi_{kR} + f_N \psi_{jL} \psi_{kL}) (\psi_{j'L} \psi_{k'R} + \psi_{j'R} \psi_{k'L}) + \right. \\
&\quad \left. (\psi_{jL} \psi_{kR} + \psi_{jR} \psi_{kL}) (\psi_{j'L} \psi_{k'L} + f_N \psi_{j'R} \psi_{k'R}) \right), \\
\{Q_{ijk}^+, Q_{i'j'k'}^-\} &= 0.
\end{aligned} \tag{B15}$$

We also assume that the Q_{ijk}^\pm algebra is still valid in Gross-Neveu models as a result of

their asymptotic freedom.

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